NATURAL GEOMETRY.

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At a time when the desirability and importance of imparting technical education to all classes are generally admitted, an effort to render such education easier to both teacher and pupil is worth consideration. Intelligent reasoning is the basis of all such education. And of such reasoning mathematical is the most important, and perhaps the most difficult, to the young and uneducated.

There are two ordinary methods of learning mathematics: one, the Euclidian, which follows a road to a goal that the traveller does not see until he arrives at it; and the other, the method of most books on arithmetic and mensuration, which shows the goal without pointing out the road that leads to it. The Euclidian mode is wearisome to the young pupil, as he cannot see the use of proving in the abstract certain mathematical truths, even if his reasoning powers are sufficiently trained to do more than learn the propositions by rote. The idea that this is all that his reasoning powers are capable of doing has led the authors of arithmetical manuals to content themselves with giving him dry rules without troubling to prove to him the correctness of these rules. I propose to bring under your consideration to-night a third method, in which the pupil can see the end of the reasoning from the beginning, and can recognise the utility and necessity of each step that has to be taken to logically arrive at that end. I do not wish to decry the Euclidian method, but to explain and vindicate it to untrained minds—to show by a train of concrete reasoning, visible and tangible, that geometry is not a mere course of dry abstract induction, but a logical series of steps, each one of which it is necessary to climb in order to reach stages of truth yet in advance.

There is no novelty in applying concrete reasoning to the solution of mathematical propositions. It is not only allowable, but necessary. Euclid himself used it perforce, though as little as possible. But seeing that he was driven to use it so early as in the fourth proposition of the first book of his Elements, it may be said that all his work is partly founded upon concrete reasoning. Many distinguished mathematicians have advocated the more extensive use of this palpable reasoning wherever it was practicable,—notably the late Dean Cowie, formerly Mathematical Professor at Woolwich. And it is a remarkable fact that we owe one of the most striking and beautiful concrete demonstrations—that of determining the volume of the pyramid—to a blind man, Sir Isaac Newton's friend, Dr. Sanderson, who can only have arrived at this concrete demonstration by abstract reasoning.

The method I am about to bring under your notice was chiefly elaborated by my friend Monsieur Edouard Lagout, a distinguished engineer in the service of the French Government, and has been very extensively adopted in the technical schools connected with the War Department and the Ministries of Agriculture and Public Works, as well as in the primary and normal schools. To explain it I have had certain models prepared. And I must ask you to excuse my beginning by showing you some very simple self-evident demonstrations, as in describing the system it is necessary to begin with its elements—with the simplest problems.

You will see that the models would themselves explain most of the definitions necessary to be understood, so I need not occupy time therewith. And furthermore, we will to-night, if you please, take for granted the usual axioms.

PLANE GEOMETRY.

If we take twelve separate square inches as shown by these models and arrange them together in three rows as in Figure 1, it is evident that they form a rectangular oblong or parallelogram with an area of twelve square inches, for we can count the square inches that exactly cover the parallelogram.

Again, if we arrange nine of them in three rows, as in Figure 2, it is evident that they form a rectangle of an area of nine square inches, for we can count the square inches that exactly cover it.

It is equally evident that we can tell the area of any other rectangle we make by arranging any number of these square inches by counting them. But if we made the rectangle of a great number it would take a long time to count them, so that we should find it easier to count the number of rows there are in the rectangle and the number of square inches in each row and to multiply the two numbers together ; for we find that this will give the same number as the counting of all the square inches will. Thus, in the first figure there are three rows with four in each row, and three times four are twelve, the same number as we counted ; and in the second figure there are three rows with three in each row, and three times three are nine, which is also the same number as we counted. In like manner, however, we arrange a rectangle we shall find that this multiplication will give us the area.

We can now go a step further. We see that each square inch is an inch long and an inch high, that is, each side of it is an inch long. So it is evident that when we put four of them in a row the bottom line of the row is four lineal inches long, and that when we put three rows in height the side line is three inches high. And as this *lineal* measure of length and height represents the rectangle of Figure 1, whose area was twelve square inches, we find out these two things:—First, that as four *lineal* inches multiplied by three *lineal* inches give this area of twelve *square* inches, *square* measure is the product of the simple multiplication of two *lineal* measures; and secondly, that we need not count the rows of square inches in a rectangle, nor the number of them in each row, but by simply measuring with a rule the length and height of it, and multiplying these together, we shall get the area of the rectangle. Thus we have proved the correctness of the rule that multiplying the length by the height will give the area of a rectangle.

We have also illustrated some other things. For example: if the rectangle be a square like Figure 2, it is evident that in multiplying the length by the height we were multiplying a number by itself—three by three. As this produces a square, multiplying a number by itself is called squaring it, and the product of a number multiplied by itself is called its square. Thus, nine is the square of three, twenty-five of five, and so on. Again, as a square of an area of nine inches is based upon a line of three inches, and is, as it were, grown upon it, three is called the square root of nine, and in like manner five is the square root of twenty-five, and so on.

To revert to the rule for obtaining the area of a rectangle. We proved its truth by applying square inches, which for the present we have assumed as our standard measure, actually to the rectangle. But as all our standards for measuring areas are squares—square inches, square feet, square miles, and so on—it is evident that we cannot in like manner prove the correctness of a rule for measuring areas that are not rectangular. For instance, we cannot cover Figure 3 with square inches; either some parts of the figure will be uncovered, or some parts of the square inches unoccupied. How, then, can we be sure that we are right when we say that such an irregular figure contains so many square inches?

Let us take a rectangle like Figure 4, and call it rectangle A. If it is eight inches long and four inches high, we know that its area is $8 \times 4 = 32$ square inches. Then let us divide it into a square and two triangles by these models arranged as in Figure 5; it is evident that the base of the square is four inches, for its height is four inches, and therefore that its area is $4 \times 4 = 16$ square inches. It is also evident that the base and height of each of the triangles is four inches. The square and the two triangles together cover the whole of the rectangle A, and therefore the sum of their areas is equal to the area of the rectangle. Now let us take another rectangle, B, equal to A, and the same two triangles, and arrange them therein as represented in Figure 6 with this sloping parallelogram between them ; the sloping parallelogram and the two triangles together cover the whole of the rectangle exactly, just as the square and the triangles did. If we now from the equal rectangles take the equal triangles, the remainders must be equal, that is, the sloping parallelogram is equal to the square. Wherefore the area of the sloping parallelogram is 16 square inches. But the base and vertical height of the sloping parallelogram are each four inches, as the base is the difference between the base of the rectangle and that of one of the triangles, $8 - 4 = 4$; and the vertical height of it is the height of the rectangle. And as multiplying this base by this height, four by four, make 16, it is evident that the area of this sloping parallelogram is to be found by the multiplication of its base by its height.

It is evident that whatever shaped parallelogram we may take, we can form a rectangle by applying two equal right-angled triangles to it as shown in the upper part of Figure 7. If we then arrange the two triangles as shown in the lower part of Figure 7, it is evident that we leave a smaller rectangle, that must have a base, height, and area each respectively equal to those of the sloping parallelogram in the upper part of the figure, and that these equal areas are the product of the multiplication of the equal bases and heights. Consequently we can extend the rule we have proved, and say of any parallelogram, whether rectangular or not, that its area is equal to its base multiplied by its height; and have shown that we can apply square measure to the measurement of things that are not square. But it is to be noted how square measure asserts itself by demanding that the base and height shall be measured at right angles, or, popularly speaking, *squarely* to each other.

We can extend the principle we have thus established. If two equal triangles, such as the models we have been using, be placed together, so that one side of one coincides with the equal side of the other, and the other equal sides are opposite to each other, it is evident that they form a parallelogram, for the opposite sides are equal. Thus these equal right-angled triangles may be arranged to form either the parallelogram shown in Figure 5, or that shown in Figure 8. And these equal scalene triangles may be arranged to form the parallelograms shown in Figures 9, 10, or 11. Every triangle may thus be regarded as half of a possible parallelogram. In every parallelogram thus formed of two triangles, it is evident that each triangle has the same base and vertical height as the parallelogram, and half its area; and equally evident that, if multiplying its base by its whole height will give the area of the whole parallelogram, multiplying its base by half its height will give half its area, that is, the area of the triangle. Wherefore the area of a triangle is equal to its base multiplied by half its height.

As triangles can be thus measured, all plane surfaces bounded by straight lines can be measured, for they can all be divided into triangles. Thus a field of the shape of Figure 12 can be divided as shown, and then the area of each triangle being found, the sum of these areas will be the area of the field.

The power we thus possess of measuring triangles, is the aim and goal of all geometry. The angles and the areas of other figures cannot be ascertained by measuring their sides. Four or any greater number of lines of any given lengths may describe figures of infinitely varied shapes and areas. Thus these four lines, each of four inches in length, may describe a square of 16 square inches in area, as shown in Figure 13, or a parallelogram of any smaller area, one of which is shown by Figure 14. But these three lines, each four inches long, can only describe the one shape and area shown in Figure 15. To alter its shape or area, the length of one or all of its sides must be altered, consequently the lengths of the sides of a triangle fix its shape—that is, determine its angles also; but the lengths of the sides of a polygon do not. So the dividing of a polygon into triangles ties the figure into a fixed shape, and therefore, in surveying, the diagonal lines that do so are often called *tie* lines.

By these concrete, visible demonstrations, we have now proved the truth of rules that enable us to measure any accessible plane bounded by straight lines. But very often there are parts of planes that are so inaccessible that we cannot apply our measures to them. How, then, are we to measure them, and how are we to be sure that our measurements are exact?

I cannot occupy your time, nor pretend within the limits of this paper to give the complete course of Geometry that is involved in the answer to this question. But to show how much this course can be facilitated by concrete methods, let me visibly demonstrate the 47th proposition of the 1st Book of Euclid's Elements without having to previously demonstrate all the preceding propositions that lead up to it.

Let us take two equal squares A, and B, Figure 16, and four equal right-angled trangles F, G, H, and I, and arrange them as shown so as to leave a square C inscribed in A. The square C and the four triangles occupy the whole of square A. Now let us take the same four equal right-angled triangles and arrange them in the two parallelograms upon square B, leaving two squares D and E. The two squares D and E and the four triangles occupy the whole of square B. If from the equal squares A and B we take the equal right-angled triangles the remainders are equal, therefore the square C must be equal to the sum of squares D and E.

But the square C is the square of the hypotenuse of each of the equal right-angled triangles F, G, H, and I, and the square D is the square of one of the sides containing the right angle, and E is the square of the other side containing the right angle. Wherefore the square of the hypotenuse of these right-angled triangles is equal to the sum of the squares of the sides containing the right angle. And as it is evident that the same result would follow whatever shaped right-angled triangles we took to arrange as we have shown—and in practice I would make students use other shaped ones—it follows that in all right-angled triangles the square of the hypotenuse is equal to the sum of the square of the other sides.

The truth thus demonstrated is the basis of all trigonometry, and by it we can not only exactly measure the inaccessible, but we can also sufficiently exactly measure the incommensurable. It is beyond the scope of this paper to show how this is done. But to illustrate how concrete demonstrations may be used to simplify the proofs of measurement of the incommensurable, let us take the rule for ascertaining the area of a circle by multiplying half the circumference by the radius.

If the circle (Figure 17) be divided into 24 equal parts, as in this model, half these parts may be arranged with their points one way and half with their points the other (one of these parts being again divided into two). The circle has now assumed a rectangular shape (Figure 18), the length of which is half of the circumference, and the height of which is the radius. The bases of the triangles are not perfectly straight lines, but had the circle been divided into hundreds or thousands of triangles instead of twenty-four, the eye could not have detected the little curves. And as the calculated length of the circumference is taken in the rule it may be held to be exact.

SOLID GEOMETRY.

However useful concrete demonstrations may be in Plane Geometry they are greatly more so in Solid Geometry, as it is always difficult to explain—to beginners at least—the figure of a solid projected upon paper. This is especially the case when solids of one form have to be divided into solids of other forms for purposes of measurement or of demonstration.

You will readily see, without my occupying time by actual demonstration, how, by forming rectangular parallelepipeds with these cubic inches, we can prove that multiplying their length by their breadth will give their base, and multiplying this base by their height will give their volume, in a manner analogous to that employed with square inches to form and measure rectangles :—how we can thus illustrate that cubic measures are the result of the double multiplication of lineal measures, just as square measures are the result of their single multiplication :—how we can show the meaning of the terms *cube* and *cube root* as we did that of *square* and *square root* :—how we can extend the rule for measuring rectangular parallelepipeds to all parallelepipeds, by dividing rectangular ones into two equal prisms as in these models (Figure 19), and then arranging the prisms as in Figure 20, to form a sloping parallelepiped, wherein it is evident that the base, height, and volume of the solid are unchanged however much its shape may be :—and how, by this division of parallelepipeds into prisms we may prove that the volume of the prism is equal to one of its rectangular sides as base multiplied by half its height.

As all planes bounded by straight lines may be divided into triangles, so all solids bounded by such planes may be divided into pyramids. Therefore if we can measure pyramids we can measure all such solids. That the formula for measuring pyramids—volume = base $\times$ $\frac{1}{3}$ height, is correct may be proved by concrete demonstrations of various kinds, as, for instance, this one based on the beautiful one imagined by blind Dr. Sanderson, that is before alluded to.

Let us take the two cubes represented by these models (Figures 21 and 22) and say they are each three inches in length, breadth, and height. They are consequently equal to each other, each having a volume of 27 cubic inches, and their basis are equal to each other, being each $3'' \times 3'' = 9$ square inches. If one be now divided into six equal pyramids (Figure 21), and the other into six equal layers (each a parallelepiped) as in Figure 22, it is evident that each pyramid is equal to each layer, for each is respectively the sixth part of equal cubes, that is, each has a volume equal to $\frac{27}{6} = 4\frac{1}{2}$ cubic inches. This is evidently the volume of each layer, for the height of each is $\frac{3}{6} = \frac{1}{2}$ an inch, and the base $9'' \times \frac{1}{2} = 4\frac{1}{2}$. But as the base of the pyramid is also $9''$ it must also be multiplied by $\frac{1}{2}''$ to produce its volume of $4\frac{1}{2}$ inches. As the height of two pyramids is equal to that of the cube, this height is $\frac{3}{2} = 1\frac{1}{2}''$ and $\frac{1}{2}''$ is one third of this. Wherefore the volume of one of these pyramids is equal to its base $\times \dfrac{\text{height}}{3}$.

Again, by dividing a cube into three irregularly-shaped pyramids (Figure 23), as shown by these models, each pyramid has the same base and height as the cube, but only a third of its volume. As multiplying the base by the height gives the volume of the cube, so multiplying its base by a third of its height must give a third of its volume, that is, the volume of one of the equal pyramids.

I will not further occupy your time by ocular demonstration that the formula holds good whatever the form of the pyramid, but will conclude with a practical exhibition of the utility of the system.

When materials of any description, such as corn, or gravel, or broken stone are formed into regular heaps for measurement, they are usually put into a shape more or less like a truncated pyramid. Suppose that these models (Figure 24) be a heap of sand to be measured, and that the heap is 12 feet *long* at the bottom and 8 feet at the top; 8 feet *broad* at the bottom and 4 feet at the top ; and 3 feet *high* measured perpendicularly : what is the volume of it ?

One or other of two methods is usually employed to find this out. The first is to multiply the mean length by the mean breadth, and the product by the height. From the dimensions we have assumed, the mean length is 10 feet and the mean breadth 6 feet, and the height 3 feet. Therefore, $10 \times 6 \times 3 = 180$ cubic feet for the volume of the heap.

The other method is—Multiply the mean base by the height. From the above dimensions the lower base is $12 \times 8 = 96$ square feet, and the upper $8 \times 4 = 32$ square feet. The mean base is therefore $\dfrac{96 + 32}{2} = 64$ square feet, which, multiplied by the height, 3 feet, gives 192 cubic feet as the volume of the heap. Both these cannot be right. Is either ?

Let us divide the heap into shapes that we know how to measure. There is the central parallelepiped 8 feet long, 4 feet broad, and 3 feet high. Its volume, therefore, is $8 \times 4 \times 3 = 96$ cubic feet. There are two long prisms each 8 feet long, 2 feet wide, and 3 feet high, which being put together as in Figure 19, make a parallelepiped whose volume is $8 \times 2 \times 3 = 48$ cubic feet. There are two short prisms each 4 feet long, 2 feet broad, and 3 feet high, which also being placed together form a parallelepiped whose volume is $4 \times 2 \times 3 = 24$ cubic feet. There are four pyramids, whose bases are each 2 feet by 2 feet $= 4$ square feet, and whose height is 3 feet. The volume of each is therefore $4 \times \frac{3}{3} = 4$ feet, or 16 cubic feet for the four pyramids. These different solids make up the entire heap, whose volume is therefore $96 + 48 + 24 + 16 = 184$ cubic feet.

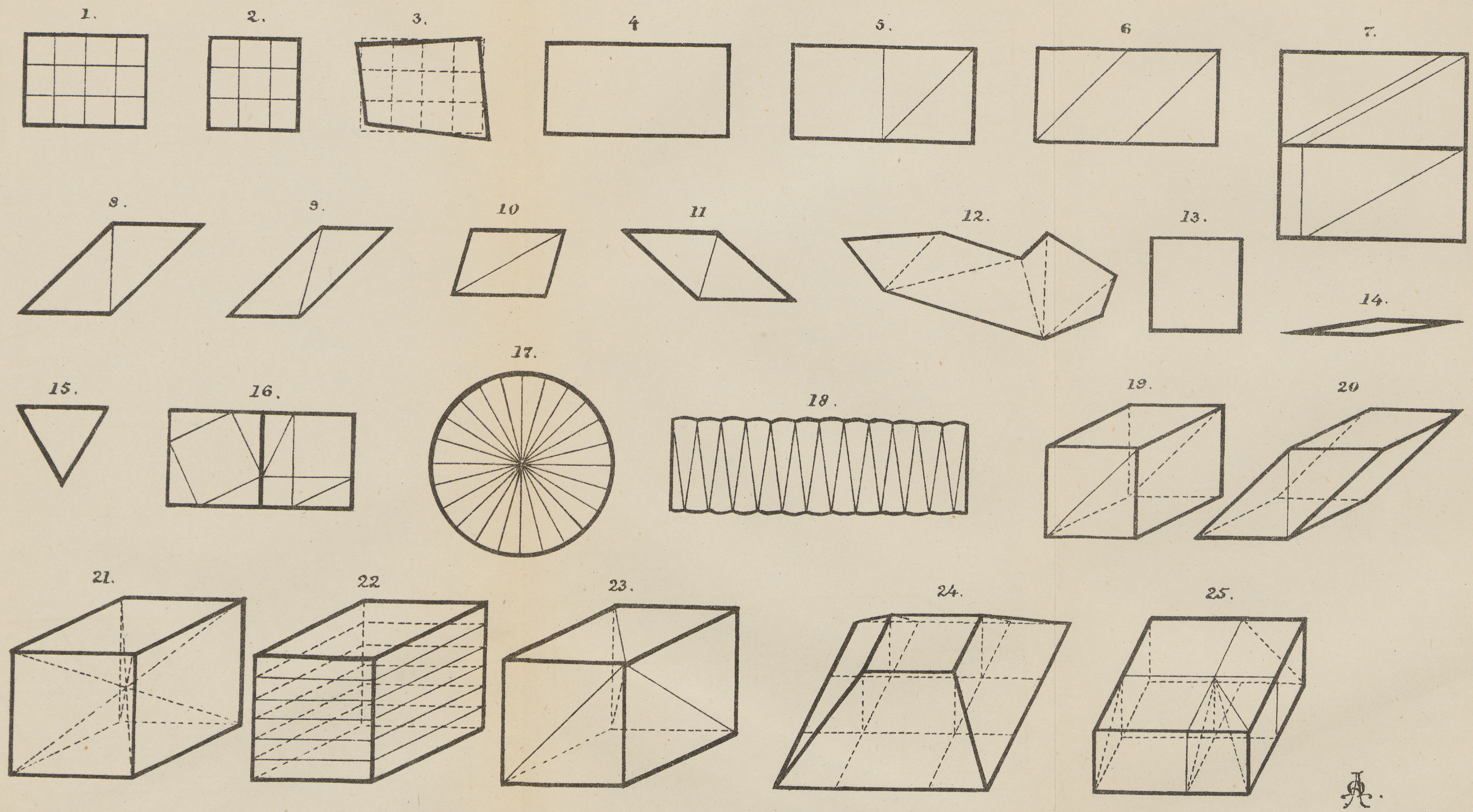
But the error in the two systems usually adopted could have been shown in the concrete without any calculations. To take the first method: Mean length × mean breadth × height. The top length is the length of a long prism, and the bottom length is the length of the long prism and that of the bases of two pyramids, therefore the mean length is that of the long prism and of the base of one pyramid. In like manner it is clear that the mean breadth is the length of a short prism and of the base of a pyramid. If, therefore, this method be right, the heap is equal to a parallelepiped of this mean length and breadth, with a height equal to that of the heap. The models arranged as in figure 25 show such a parallelepiped, and it is made up of the central parallelepiped of the heap, the two long and the two short prisms arranged as in figure 19, and *three* pyramids arranged as in figure 23. But the heap had *four* pyramids; the parallelepiped therefore is not equal to the heap, and the method is consequently incorrect.

So also is the second method—Volume equal to mean base multiplied by height. The top base is that of the central parallelepiped of the heap. The bottom base is that of the same solid, together with those of two long and two short prisms and four pyramids. The sum of these is twice the base of the central solid, of the long prism, of the short prism, and four times that of a pyramid, so the mean base will be that of the central solid, a long prism, a short prism, and two pyramids, and this should be the base of the parallepiped of the height of the heap, whose volume should (if this method be right) equal that of the heap. That is, it would be necessary to add to figure 25 the base of another pyramid and complete the parallelepiped upon that base up to the height of the rest. To do this two more pyramids would be required, but all the constituent parts of the heap are already used, and consequently this method is wrong also.

The first method was wrong by the omission of the volume of one pyramid. This volume is equal to its base multiplied by a third of its height. Its base is equal to the difference between the extreme and mean lengths and breadths of the heap; therefore, if these differences be multiplied together, and their product by a third of the height, and this be added to the product of the first method, the true contents of the heap will be found.

These are a few illustrations of a method I venture to recommend for the preliminary teaching of mathematics. You will observe that very often it strikingly shows the intimate relations that exist between the sciences of quantity. I would not in the first place precede these simple demonstrations by any definitions. Where necessary when a scientific word is first used I would define it, but as the definition would then be given with the thing defined bodily before the eyes of the learner, it would be more easily learnt and understood, and more indelibly fixed in the mind. But generally it is better at first to trust to the general knowledge possessed by most learners, however young, of the names of elementary forms when they see them, rather than add to the bulk of material set forth as having to be learnt.

But, to my mind, the greatest advantage of the system is that it renders the understanding of mathematical reasoning so easy as also to render easy the understanding of all other logical reasoning. It visibly puts before the eye the lines upon which all reasoning proceed. It thus increases the usefulness of mathematics as a promoter of exactness in reasoning, and assists learners in all their other studies by giving them clearer and larger powers of apprehension. It not only simplifies geometry, but makes the acquirement of all other science more easy.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